2023 CMWMC Individual Round Solutions

1. Compute
$$\tfrac{1}{2}\left(1+\tfrac{1}{2}\left(1+\tfrac{1}{2}\left(1+\tfrac{1}{2}\left(1+\tfrac{1}{2}\left(1+\tfrac{1}{2}\right)\right)\right)\right)\right).$$

   Proposed by Connor Gordon

   Answer. $\frac{63}{64}$

   Solution. You simply compute from the outside in, or distribute to get $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}=\boxed{\frac{63}{64}}$.

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2. Let p, q, r be primes with $p+q+r=26$. What is the maximum possible value of pqr?

   Proposed by Ishin Shah

   Answer. 286

   Solution. We have that one of these primes must be 2 since they add to an even number.

   This means the remaining primes add to 24.

   We maximize the product of the remaining primes by making them as close to their average of 12 as $(12-d)(12+d)=144-d^2$ and we want d to be small.

   Thus, since $d=1$ works and $d=0$ doesn't, we get our primes are $2, 11, 13$ which gives a product of $\boxed{286}$.

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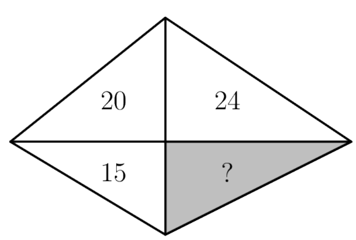
3. In the diagram below, the areas of three triangles are given. What is the area of the shaded triangle?



   Proposed by Lohith Tummala

   Answer. 18

   Solution. Since the top two triangles have the same height with respect to the horizontal line, we could find the ratio of the bases as the ratio of the areas. We could do this on the bottom as well.

Thus, both ratios are the same, so we get $\dfrac{24}{20} = \dfrac{?}{15}$. This gets $? = \boxed{18}$

4. Shyla wrote the letters C, M, W, M, C on five otherwise identical marbles, put them in a bag, and shuffled them up. She then pulls out the five marbles one-by-one in some order. What is the probability that she pulls the marbles in the order C, M, W, M, C?

   (Keep in mind that Shyla's M, turned upside-down, looks like a W and vice versa.)

   Proposed by Lohith Tummala

   Answer. $\frac{1}{10}$

   Solution. The first marble and the last marble must be a C. The middle three can be any order of M and W since they can be rotated to the correct orientation. Thus, out of the ten possible ways to arrange 2 C's and 3 W's (can be M's), one of them is in the order CWWWC. Thus, we have $\boxed{\dfrac{1}{10}}$.

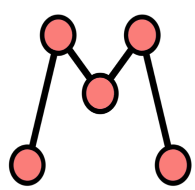
5. Find the remainder when 7^7 is divided by 100.

   Proposed by Justin Hsieh

   Answer. 43

   Solution. We know that $7^4 = 2401$. So, 7^7 has the same remainder as $7^3 = 343$ when divided by 100, so our answer is $\boxed{43}$.

6. Consider the M of the CMIMC logo. Imagine that you play a game: you start at the middle node. On each turn, you have a 50% chance of moving to either neighbor. However, if at any time you only have one neighbor, the game ends.



   What is the probability that the game is still going after ten moves?

   Proposed by Lohith Tummala

   Answer. $\dfrac{1}{32}$

   Solution. After one move, we are not done, since we are necessarily at an upper node. In our second move, there is a half chance that we end at a bottom node (and are done), and a half chance that we end up in the middle node. Thus, after two moves, there is a $1/2$ chance that the game is done, and a half chance that we are back at the middle node. Thus, if we are not done, then we reset to the original state.

We continue this pattern. After four moves, there is a $\frac{1}{4}$ chance that we are not done. After six moves, there is a $\frac{1}{8}$ chance. After eight moves, 1/16 chance. After ten moves, we are at $\boxed{\frac{1}{32}}$.

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7. Charlotte travels at a constant speed of 2 miles per hour up a mountain, and at a constant speed of 5 miles per hour back down. Written as a simplified fraction, what is Charlotte's average speed over the whole trip?

   Proposed by Corey Predella

   Answer. $\frac{20}{7}$

   Solution. WLOG, assume the distance up the mountain is 10 miles. Therefore, the time it takes to travel up the hill is given by $\frac{2 \cdot 5}{2} = 5$ hours and the time it takes to travel down the hill is $\frac{2 \cdot 5}{5} = 2$ hours. The total distance and time of the trip is thus $2 \cdot 10 = 20$ miles and $2 + 5 = 7$ hours. Then our average speed is $\boxed{\frac{20}{7}}$.

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8. Jenny is trying to take a photo of the CMIMC club members. They first try to put everyone in rows of 5, but there are 2 people left over. Then they try to put everyone in rows of 7, but there are 3 people left over. Then they try to put everyone in rows of 9, but there are 4 people left over. Given that there are fewer than 300 people in CMIMC, how many club members are there (excluding the photographer)?

   Proposed by Alan Abraham

   Answer. 157

   Solution.

   Let's say x is the number of people. We know $x \equiv 2 \mod 5, x \equiv 3 \mod 7$, and $x \equiv 4 \mod 9$. We know by CRT there will be a unique solution $\mod 315$ since $315 = \text{lcm}(5, 7, 9)$. One can brute force search, but an easier way is by noting that $x \equiv (-\frac{1}{2}) \mod 7, \mod 9, \mod 13$, so $x \equiv (-\frac{1}{2}) \mod 315$ will be the solution. So our answer $(-\frac{1}{2}) \equiv (314/2) \equiv \boxed{157} \mod 315$

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9. Simplify the following fraction:

$$\frac{1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + 13 + 14 + \cdots + 298 + 299}{3 + 6 + 9 + 12 + 15 + 18 + \cdots + 297 + 300}$$

   Proposed by Henry Zheng

   Answer. $\frac{200}{101}$

Solution. Add 1 to the expression. Then we have

$$\frac{1+2+\cdots+300}{3+6+9+12+15+18+\cdots+297+300}.$$

Using the addition summation formula we have $\frac{300\cdot301}{3\cdot100\cdot101}=\frac{301}{101}$. Subtracting 1 we have $\boxed{\frac{200}{101}}$.

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10. How many ways can we arrange the numbers $1,1,2,2,3,3$ such that 1 is not in the third position?

Proposed by Ishin Shah

Answer. 60

Solution. There are $\frac{6!}{(2!)^3}=90$ ways to arrange the numbers, and $\frac{1}{3}$ of them have a 1 in the third position. So $90-\frac{1}{3}\cdot90=\boxed{60}$ ways work.

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11. Let ABC be a triangle with $AB=5$, $BC=12$, $AC=13$. We put points D_1,D_2 on AB, points E_1,E_2 on BC, and points F_1,F_2 on AC such that:

- $AD_1=BD_2=\frac{5}{3}$,
- $BE_1=CE_2=4$,
- $AF_1=CF_2=\frac{13}{3}$.

Find the area of the hexagon $D_1D_2E_1E_2F_1F_2$.

Proposed by Ishin Shah

Answer. 20

Solution. The area of ABC is 30. Then, BD_2E_1, E_2F_1C, and D_1AF_2 are similar to ABC, but $\frac{1}{3}$ of the side length, making them each $\frac{1}{9}$ of the area. This means we get $30-\frac{1}{9}\cdot3\cdot30=\boxed{20}$.

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12. The polynomials

$$P(x)=x^3-4x^2-25,\quad Q(x)=x^3-8x^2+11x+20$$

share a common root. What is this root?

Proposed by Henry Zheng

Answer. 5

Solution. Subtract the two polynomials from each other. We get $4x^2-11x-45$, which applying the quadratic formula, gets us $x=(x-5)(4x+9)$. Substituting back in, we see only 5 works, giving us $\boxed{5}$ as the answer.

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13. What is the smallest positive integer for which the product of its factors is a multiple of 2024? (For example, the product of the factors of 12 is

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 12 = 1728,$$

but this is not a multiple of 2024.)

Proposed by Lohith Tummala and Jenny Quan

Answer. 506

Solution. Let the product of the factors of x be N. $2024 = 2^3 \cdot 11 \cdot 23$, so the primes 2, 11, and 23 have to show up in x. Therefore, we have 1, 2, 11, and 23 as factors of x. However, this means that their LCM, which is 506, must also be a factor of x. Adding in the factors of 506, which are necessarily factors of x, we have 1, 2, 11, 22, 23, 46, 253, and 506. These are all the factors of 506, and their product indeed divides 2024, so our answer is $\boxed{506}$.

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14. A parallelogram has sides of length 13 and 15. If one diagonal of this parallelogram has a length of 14, find the length of the other diagonal.

Proposed by Connor Gordon

Answer. $4\sqrt{37}$

Solution. Split the parallelogram into two 13-14-15 triangles, each of which can in turn split into a 5-12-13 right triangle and a 9-12-15 right triangle. Dropping perpendicular from the far vertices onto the diagonal of length 15, we can form a right triangle with short leg 4, long leg 24, and hypotenuse the other diagonal of the quadrilateral. This then gives us a length of $\boxed{4\sqrt{37}}$.

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15. Sharon is playing a variant of chess where the initial pieces on the first row are shuffled. There are 1 king, 1 queen, 2 knights, 2 rooks, and 2 bishops, and pieces of the same type are indistinguishable from one another. Furthermore, bishops must be on opposite-colored squares, and both sides have the same configuration. How many possible starting positions are there?

Proposed by Ishin Shah

Answer. 2880

Solution. We have $\binom{4}{1}$ ways to choose a black-squared bishop and $\binom{4}{1}$ ways to choose a white-squared bishop. For the remaining 6 pieces, there are $\frac{6!}{2 \cdot 2}$ ways to place them on other 6 squares, since we double count rooks and knights.
Together, we have $\binom{4}{1}\binom{4}{1}\left(\frac{6!}{4}\right) = \boxed{2880}$.

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16. Claire digs a circular hole of radius $r < 4$ into perfectly-flat ground. If she places a sphere of radius 4 into the hole, its lowest point sits at some distance d below the surface. If she instead places a sphere of radius 8 into the hole, its lowest point sits at a distance $d - 2$ below the surface.

Compute the radius of a sphere such that, if it were placed into the hole, its center would sit at a distance of $d-1$ below the surface.

Proposed by Connor Gordon

Answer. $\frac{19}{4}$

Solution. Dropping a perpendicular from the center of each sphere to the hole, we get the distance from the center of the radius 4 sphere to the hole is $\sqrt{16-r^2}$, and thus the lowest point sits at a distance $4-\sqrt{16-r^2}$ below the surface. Similarly, the highest point sits at a distance $8-\sqrt{64-r^2}$ below the surface.

Subtracting these gives $2=(4-\sqrt{16-r^2})-(8-\sqrt{64r^2})$, or $\sqrt{64-r^2}-\sqrt{16-r^2}=6$. Solving this using your favorite method gives $r^2=15$, so $r=\sqrt{15}$ and thus $d=3$.

Letting our desired sphere have radius R, we want $R-\sqrt{R^2-15}=2$. Solving this using your favorite method yields $R=\boxed{\dfrac{19}{4}}$.

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17. For $x \in \{0,1,2,\ldots,10000\}$, let

$$f(x)=\left\lfloor \frac{x^2}{2024} \right\rfloor.$$

How many distinct values does f take as x ranges over $\{0,1,2,\ldots,10000\}$?

Proposed by Ishin Shah

Answer. 9495

Solution. If $\frac{(x+1)^2}{2024}>\frac{x^2}{2024}+1$, then x and $x+1$ have different values. This is true for $x \geq 1012$. Thus, each value of x from 1012 to 10000 can make different values, giving 8989 solutions.

Now every value from 0 to $\lfloor \frac{1012^2}{2024} \rfloor = 506$ could work if $\lceil \sqrt{2024a} \rceil$ with some value of a is different than with every other value of a as this makes a the value that works. This must be true if $\sqrt{2024(a+1)} \geq \sqrt{2024a}+1 \implies 2024a+2024 > 2024a+1+2\sqrt{2024a} \implies \frac{2023^2}{4 \cdot 2024} > a$. This means every number from 0 to 505 works, adding 506 more possible choices.

Thus, the answer is $506+8989=\boxed{9495}$.

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18. Find the greatest common divisor for F_{555} and F_{550}, where F_n is the nth term in the Fibonacci sequence with $F_1=F_2=1$.

Proposed by Henry Zheng

Answer. 5

Solution. Notice $F_{555}=11F_{550}+5F_{549}$. Notice adjacent terms in the Fibonbacci sequence are relatively prime. Thus, we can see that 5 divides F_{555} by mod counting, and therefore, we're done as the answer is $\boxed{5}$.

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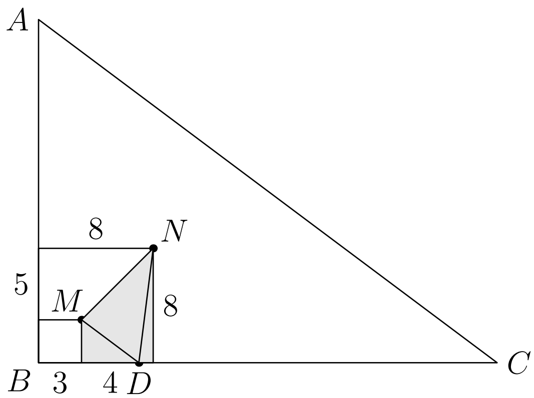
19. Given a triangle $\triangle ABC$ such that $AB = 24, BC = 32, AC = 40$. Let D be a point on BC such that $BD = 7$. Let N be the incenter of $\triangle ABC$ and M be the incenter of $\triangle ABD$. What is the area of $\triangle NDM$?

Proposed by Lohith Tummala

Answer. $\frac{35}{2}$

Solution.



We take the area of the shaded trapezoid and subtract both triangles to get the area we want. This is $\dfrac{55}{2} - 6 - 4 = \boxed{\dfrac{35}{2}}$.

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20. Suppose a, b, c, d are uniformly chosen divisors of 6^5. How many possible ways are there to choose such a, b, c, d such that

$$a \mid b, \quad a \mid c, \quad b \nmid c, \quad c \nmid b, \quad c \mid d, \text{ and } b \mid d?$$

Note $a \mid b$ is shorthand for a divides b. For example, $2 \mid 12$.

Proposed by Ishin Shah

Answer. 9800

Solution. Let $a = 2^{w_1}3^{w_2}, b = 2^{x_1}3^{x_2}, c = 2^{y_1}3^{y_2}, a = 2^{z_1}3^{z_2}$.

We have $0 \leq w_1 \leq x_1, y_1 \leq z_1 \leq 5$ and $0 \leq w_2 \leq x_2, y_2 \leq z_2 \leq 5$ and one of the following: $x_1 < y_1, x_2 > y_2$ or $x_1 > y_1, x_2 < y_2$. Thus, we must have $0 \leq w_1 \leq x_1 \leq y_1 \leq z_1, 0 \leq w_2 \leq y_2 < x_2$ or $0 \leq w_1 \leq y_1 \leq x_1 \leq z_1, 0 \leq w_2 \leq x_2 < y_2 \leq z_2$.

Then, we could use the prime version of each variable above to change all the $\leq$ to $<$ and change the upper bound to 7. This means that for any two given choosing of 4 numbers from 0 to 7, we have 2 solutions. This makes the answer $\binom{8}{4}^2 \cdot 2 = \boxed{9800}$

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21. (Estimation) Nine points are placed in the plane such that no three are collinear. Estimate the minimum number of convex quadrilaterals which can possibly be formed by connecting four of the points.

Proposed by Ari Battleman

Answer. 36

Solution. It is derived from some research paper.